Int. J. Solids Structures Vol. 24, No. 5, pp. 439–458, 1988
Printed in Great Britain.

0020-7683/88 $3.00+.00

MODELING LAMINATED COMPOSITE STRUCTURES AS ASSEMBLAGE OF SUBLAMINATES

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(Received 31 March 1986; in revised form 29 June 1987)

Abstract—A mixed variational principle is presented for stress analyses of laminated composite structures with or without delaminations and discontinuities. Of particular interest is the use of a functional involving interlaminar stresses and in-plane strains, instead of strain or complementary energy. A laminate is then treated as an assemblage of sublaminates with assumed variations of displacements and interlaminar stresses through the thickness of each sublaminate. Application of the variational principle, at this stage, reduces the problem to a set of differential equations and constitutive relations for each sublaminate similar to those encountered in higher order plate theories, as well as continuity conditions of displacements and equilibrium of surface tractions at perfectly bonded interfaces. It is demonstrated with an example of a disbonded laminate modeled as an assemblage of two sublaminates that the mixed boundary conditions at a partly bonded interface yield concentrated forces at the disbond tips. This is the exact form of stress singularity in this model. Finite element analyses are utilized for obtaining solutions with more sublaminates. Stress fields and energy release rates at delamination tips are compared with elasticity and standard finite element solutions in another illustrative example.

INTRODUCTION

Growth of disbonds created due to imperfect bonding or foreign object impact can severely influence the load bearing capacity of laminated composites. For this reason, delamination fracture has been the subject of many investigations. Methods employing Fourier transforms and singular integral equations discussed in several studies[1–3] are suitable for stress analysis when the disbonds are located away from edges or other discontinuities. In many cases, such disbonds can, however, occur near edges or other geometric discontinuities, and may in fact originate at such locations. Free edges in laminates, ply drops, and bonded joints are examples. Rigorous elasticity solutions and numerical calculations for such problems are impossible from practical considerations. Standard finite element methods (or those employing elements with singular fields) have been utilized in such problems[4, 5]. In such currently available models with assumed displacement fields, it is necessary to model each layer of different orientation separately. In many cases, however (see results for elliptic disbonds and those for two-dimensional (2-D) problems[2, 3, 6, 7] for large disbonds), an approximate solution based on the assumption that disbonds are located between sublaminates which deform according to shear deformation plate theory yields strain energy release rates close to elasticity solutions. This is true even though the stress singularities in these approximate solutions appear as concentrated forces at disbond tips[6–8]. This is especially true when the disbond dimensions are large compared to the thickness and the plate models yield a very good approximation. Since disbonds of comparatively small sizes are usually not critical, it appears that for all practical purposes it may be unnecessary to use standard finite element techniques modeling each layer. For more accurate solutions, laminated plate theories of higher order, or a large number of sublaminates through the thickness of the laminate with each sublaminate still containing many layers, can possibly be utilized. In higher order plate theories, however, calculation of plate properties based on assumed strain fields does not accurately represent effects of some stiffnesses. For example, shear stiffnesses are more accurately represented by the assumption of constant stress fields, and correction factors having values close to those of isotropic or orthotropic plates[9, 10], than by the assumption of constant strain fields. For these reasons, in this study a mixed variational principle with assumed linear variation of displacement fields and transverse shear stresses, and constant transverse normal stress through the thickness, is

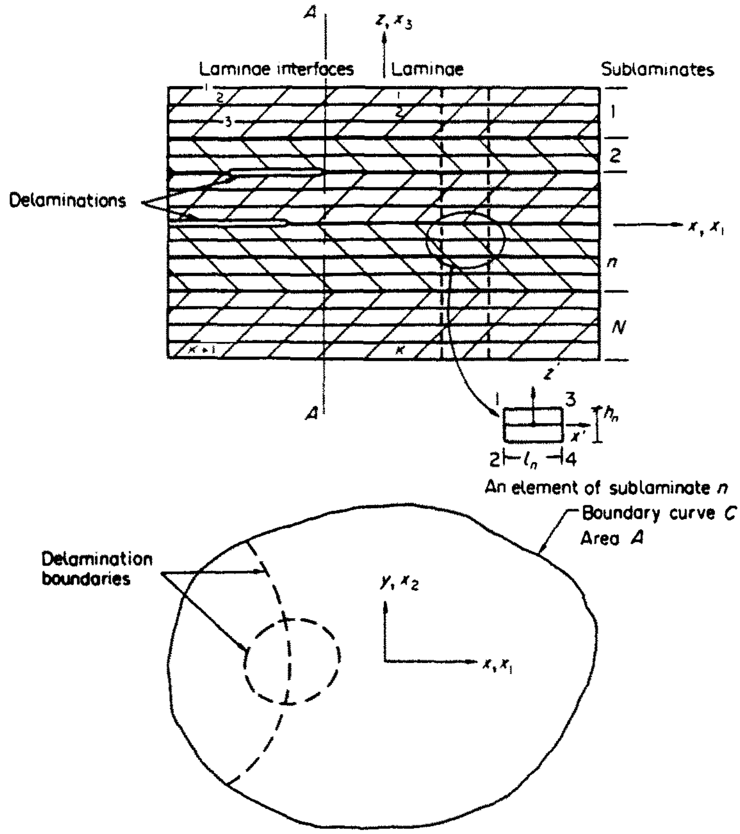


Fig. 1. A laminate divided into N sublaminates.

formulated (as described later) to obtain a higher order plate theory for each sublaminate, which is applicable to the general three-dimensional (3-D) problem with disbonds and discontinuities (Fig. 1). Displacement continuity conditions at bonded interfaces are imposed in terms of Lagrangian multipliers, which are the tractions at those locations. Existence of concentrated line forces at disbond boundaries is also demonstrated. This formulation, although similar in form to theories developed by other investigators (see Refs [11, 12] for example) is more straightforward for practical applications.

With the imposed condition that displacements of and tractions on adjacent layers on the two sides of a bonded interface be the same (opposite in sign for traction) the Euler equations for generalized plane strain (quasi-3-D or 2-D) problems can be reduced to a set of simultaneous ordinary differential equations as in Refs [11, 12]. Counting the concentrated forces at disbond or crack tips as unknowns, there exists the required number of unknowns to satisfy the boundary conditions or other required relations across disbond or crack tips. It may be noted that such conditions on displacements (not displacement derivatives) and moment or force equilibrium (not stresses) can only be satisfied. For this reason, concentrated forces exist at such locations as in similar approximate formulations[6–8, 13]. Such a system of equations for a finite (but not small) number of sublaminates is, however, not very simple, and the procedure for development of a general algorithm can be quite involved because of the possibility that some of the characteristic roots of the system of equations can be repeated[11, 12]. Also, such an approach is useful only in linear problems. For these reasons, a finite element model is developed for quasi-3-D problems based on assumed linearly varying displacement fields in the direction parallel to laminations. String stiffnesses of rectangular elements to consider effects of large deformations, as well as change in shear stiffness to model non-linear shear response, which are important in bonded joint or other problems, can then be easily incorporated into the analysis.

MIXED VARIATIONAL PRINCIPLE

Consider a laminated plate with K laminae as shown in Fig. 1. Let u_i^k, ε_{ij}^k and σ_{ij}^k denote the displacements, strains and stresses in lamina k referred to the Cartesian coordinate system shown in the figure. Let λ_i^{+k} and λ_i^{-k} ($i = 1, 2, 3$; $k = 2, 3, \ldots, K$) be the surface tractions on laminae $k-1$ and k, respectively, on surface A_k, parts of the interface k which are not delaminated and $\bar{u}_i^k$ be the displacement on the corresponding surface. For prescribed surface tractions $\check{T}_i^*$ and/or displacements u_i^* on the boundary of such a laminated plate as well as body forces b_i one can pose the following mixed variational principle:

$$\text{Variation}\,(I) = 0 \tag{1}$$

where functional I is given by

$$I = \sum_{k=1}^{K} \int_{v_k} [W^k(\varepsilon_{\alpha\beta}, \sigma_{\gamma\delta}^k) - \sigma_{\alpha\beta}^k \varepsilon_{\alpha\beta}^k + 1/2\sigma_{ij}^k(u_{i,j}^k + u_{j,i}^k) - b_i^k u_i^k]\,\mathrm{d}v$$
$$- \sum_{k=2}^{K} \int_{A_k} [\lambda_i^{+k}(u_i^{k-1} - \bar{u}_i^k) + \lambda_i^{-k}(u_i^k - \bar{u}_i^k)]\,\mathrm{d}A_k - \int_{S_\sigma} \check{T}_i^* u_i\,\mathrm{d}S - \int_{S_u} \check{T}_i^0(u_i - u_i^*)\,\mathrm{d}S. \tag{2}$$

In the above equation, $\alpha\beta$ takes values 11, 22, 12, 21, $\gamma\delta$ varies over 33, 13, 31, 23, 32 and i and j vary from 1 to 3; repeated indices indicate summation. u_i and $\check{T}_i^0$ in the last two surface integrals are equal to the displacements of laminae the surfaces of which coincide with the bounding surface and unknown tractions on S_u, respectively. S_σ (or S_u) denotes part of the bounding surface, where stresses (or displacements) are prescribed. v_j are the direction cosines of the outward normal to the bounding surface. The functional W^k also depends on the material properties of lamina k (elastic, non-linear elastic, or elastic–plastic total deformation theory) and is chosen to satisfy the constitutive relations such that

$$\frac{\partial W^k}{\partial \varepsilon_{\alpha\beta}^k} = \sigma_{\alpha\beta}^k \quad \text{and} \quad -\frac{\partial W^k}{\partial \sigma_{\gamma\delta}^k} = \varepsilon_{\gamma\delta}^k. \tag{3a, b}$$

It may be noted that if a stress component is expressed by eqn (3a) the corresponding strain does not appear in eqn (3b). If one seeks condition (1), i.e. I to be stationary permitting the functions u_i^k, $\varepsilon_{\alpha\beta}^k$ ($\alpha\beta = 11, 12, 22$), $\sigma_{ij}^k(k = 1, \ldots, K)$, $\lambda_i^{\pm k}$, $\bar{u}_i^k$ ($k = 2, 3, \ldots, K$) and $\check{T}_i^0$ to vary over appropriate domains, then the Euler equations for the variational problem are given by the relationships described by eqn (3a) and the following set of equations:

$$-\frac{\partial W^k}{\partial \sigma_{\gamma\delta}^k} = 1/2(u_{\gamma,\delta}^k + u_{\delta,\gamma}^k), \quad \sigma_{ij,j}^k + b_i^k = 0, \quad \varepsilon_{\alpha\beta}^k = 1/2(u_{\alpha,\beta}^k + u_{\beta,\alpha}^k) \quad (k = 1, 2, \ldots, K)$$

$$-\sigma_{3i}^{k-1} = \lambda_i^{+k}, \ \sigma_{3i}^k = \lambda_i^{-k}, \ \sigma_{ij}v_j = \check{T}_i^* \text{ on } S_\sigma, \ \sigma_{ij}v_j = \check{T}_i^0, \ u_i = u_i^* \text{ on } S_u \ (k = 2, 3, \ldots, K)$$
$$\tag{4a–h}$$

$$u_i^{k-1} = \bar{u}_i^k = u_i^k, \quad \lambda_i^{+k} + \lambda_i^{-k} = 0 \quad (k = 2, 3, \ldots, K \text{ on } A_k). \tag{4i,j}$$

It is clear, therefore, that if all variables are chosen to fulfill eqn (1) all equations of the boundary value problem under consideration are identically satisfied provided

$$\varepsilon_{\gamma\delta}^k = 1/2(u_{\gamma,\delta}^k + u_{\delta,\gamma}^k); \quad \gamma\delta = 33, 13, 31, 23, \text{ and } 32. \tag{5}$$

However, in the next section it is shown how approximate solutions may be obtained by treating a laminated structure as an assemblage of sublaminates, choosing linear variations of displacements and interlaminar stresses through the thickness of each sublaminate which may contain laminae of different orientation or materials, and directly using the variational principle, eqns (1) and (2). The authors are not aware of any available mixed variational principle which allows such a choice of the field variables in an assemblage of sublaminates, each of which may be inhomogeneous.

APPROXIMATE FORMULATION AS AN ASSEMBLAGE OF SUBLAMINATES

We subdivide the laminate into N sublaminates, nth sublaminate ($n = 1, 2, \ldots, N$) containing P_n numbers of laminae $\left(\text{note that } \sum_{n=1}^{N} P_n = K\right)$ all perfectly bonded to one another and assume the following displacement fields, and interlaminar ($\sigma_{33}, \sigma_{31}, \sigma_{32}$) stress fields in each sublaminate, z_n being the z-coordinate measured from the midplane of that laminate

$$u_i^n = u_i^{0n}(x, y) + z_n \psi_i^n(x, y) \tag{6a}$$

$$\begin{aligned} \sigma_{33}^n &= \sigma_4^{0n}(x, y) \\ \sigma_{31}^n &= \sigma_5^n = \sigma_5^{0n}(x, y) + z_n \sigma_5^{1n}(x, y) \\ \sigma_{32}^n &= \sigma_6^n = \sigma_6^{0n}(x, y) + z_n \sigma_6^{1n}(x, y). \end{aligned} \tag{6b}$$

We also assume that the strain fields can be derived from displacement fields (6a) by eqns (4c) and (5) and that at all perfectly bonded interfaces inside a sublaminate $\lambda_i^{\pm k}$ and $\bar{u}_i^k$ are chosen to satisfy eqns (4d), (4e) (so that eqn (4j) is also satisfied because of eqns (6b)) and (4i), respectively. We denote the surface tractions on the surfaces of sublaminate n at $z_n = \pm h_n/2$ (either prescribed in $\check{T}_i^*$ or unknown in λ_i or $\check{T}_i^0$) as $t_i^{\pm n}$. Further, the unknown surface tractions on other bounding surfaces (parallel to the z-axis) of sublaminate n where displacements are prescribed are assumed in the form given below. It may be noted that N, M, Q, R (unknown on the bounding contour) are the stress resultants used in higher order plate theories[14].

$$\check{T}_i^{0n} = \frac{\check{N}_i^{0n}}{h_n} + \frac{12 z_n}{h_n^3} \check{M}_i^{0n}; \quad i = 1, 2 \text{ and } \check{T}_3^0 = \frac{\check{Q}^{0n}}{h_n} + \frac{12 z_n}{h_n^3} \check{R}^{0n}. \tag{6c}$$

Contracted notations for interlaminar stresses are used in eqns (6b). In what follows, we will use similar contracted notions for other stresses and strains given below and omit superscript n denoting the sublaminate

$$\begin{aligned} &\sigma_{11} = \sigma_1, \quad \sigma_{22} = \sigma_2, \quad \sigma_{12} = \sigma_3 \\ &\varepsilon_{11} = \varepsilon_1, \quad \varepsilon_{22} = \varepsilon_2, \quad \varepsilon_{33} = \varepsilon_4 \\ &2\varepsilon_{12} = \varepsilon_3, \quad 2\varepsilon_{13} = \varepsilon_5, \quad 2\varepsilon_{23} = \varepsilon_6. \end{aligned} \tag{7}$$

If the relationships between stresses and strain in the lamina k are known in terms of the (6×6) stiffness matrix $[C]$, e.g.

$$\begin{aligned} \sigma_i &= C_{ij}\varepsilon_j + C_{i\beta}\varepsilon_\beta \\ \sigma_\alpha &= C_{\alpha j}\varepsilon_j + C_{\alpha\beta}\varepsilon_\beta \end{aligned} \tag{8}$$

where i, j vary over 1, 2, and 3 and α, β vary over 4, 5, and 6, then the functional W (the superscript is omitted) in eqns (2) and (3) can be expressed as

$$W = 1/2[C'_{ij}\varepsilon_i\varepsilon_j + 2C''_{i\alpha}\varepsilon_i\sigma_\alpha - S_{\alpha\beta}\sigma_\alpha\sigma_\beta]. \tag{9}$$

It should be noted that use of this functional permits the choice of interlaminar stresses

(6b) irrespective of assumed displacement fields (6a), which yields a better representation of interlaminar stresses than that obtained by assuming displacement field only, especially in the case of sublaminates containing laminae of different orientations or materials.

The elements of the three (3 × 3) matrices C', C'' and S can be calculated from the elements of C by expressing σ_i and ε_α in terms of ε_i and σ_α ($i = 1,2,3$; $\alpha = 4,5,6$) using eqns (8). In many laminates the following relations hold and will be assumed here although one can keep the generality if desired:

$$C''_{i\alpha} = 0\,(i = 1,2,3;\alpha = 5,6) \quad \text{and} \quad S_{\alpha\beta} = 0(\alpha = 4;\beta = 5,6). \tag{10}$$

Substitution of eqns (9) and (10) and the assumed displacement and interlaminar stress fields (6a) and (6b), strain fields (4c) and (5), interface tractions $\lambda_i^{\pm k}$ (satisfying eqns (4d), (4e), $k = 2,3,\ldots,P_n$ and expressed as $t_i^{\pm n}$ at other locations) as well as $\check{T}_i^{0n}$ (6c) in eqn (2) and integration with respect to z_n over the thickness h_n yields a modified expression for I in terms of the new variables introduced above and $\bar{u}_i^n$, the unknown displacements over perfectly bonded parts of the interfaces. If one now seeks I to be stationary, permitting $u_i^n, \psi_i^n, t_i^{\pm n}, \bar{u}_i^n$ to vary over appropriate domains, one obtains a set of Euler equations of the variational problem which are similar to those encountered in higher order plate theories[14]. There are, however, some differences because of the assumption of displacements as well as interlaminar stresses and the mixed boundary conditions considered here. For these reasons, we give below the final form of these equations which is suitable for practical applications. The intermediate steps are omitted here for brevity, but can be found in Ref. [15]. In the following equations N, M, Q, R are the stress resultants used in higher order theories. ε^0, κ, γ^0, γ^1, are extensional strains, curvatures, and midplane and linearly varying shear strain components, respectively (the superscript n, denoting sublaminate n is omitted). A is the plan area of the laminated structure, A_n is the bonded part of the interface n and A_n^* is a part of the unbonded portion where known displacements are prescribed. C_u^n (or C_σ^n) are parts of the boundary contour of sublaminate n where displacements (or stresses) are prescribed.

Differential equations for sublaminate n

$$\begin{aligned}
N_{1,1}+N_{3,2}+t_1^+ + t_1^- + f_1 &= 0\\
N_{3,1}+N_{2,2}+t_2^+ + t_2^- + f_2 &= 0\\
Q_{1,1}+Q_{2,2}+t_3^+ + t_3^- + f_3 &= 0\\
M_{1,1}+M_{3,2}+\frac{h_n}{2}(t_1^+ - t_1^-)+g_1-Q_1 &= 0\\
M_{3,1}+M_{2,2}+\frac{h_n}{2}(t_1^+ - t_2^-)+g_2-Q_2 &= 0\\
R_{1,1}+R_{2,2}-N_4+\frac{h_n}{2}(t_3^+ - t_3^-)+g_3 &= 0
\end{aligned}$$

$$f_i, g_i = \int b_i(1, z_n)\, dz_n. \tag{11}$$

Constitutive relations for sublaminate n

$$\begin{aligned}
N_i &= A_{ij}\varepsilon_j^0 + B_{ij}\kappa_j + A_{i4}\varepsilon_4\\
M_i &= B_{ji}\varepsilon_j^0 + D_{ij}\kappa_j + B_{i4}\varepsilon_4; \quad i,j = 1,2,3\\
N_4 &= A_{i4}\varepsilon_i^0 + B_{i4}\kappa_i + A_{44}\varepsilon_4\\
Q_\alpha &= L_{\alpha\beta}\gamma_\beta^0 + L'_{\alpha\beta}\gamma_\beta^1\\
R_\alpha &= L'_{\beta\alpha}\gamma_\beta^0 + L''_{\alpha\beta}\gamma_\beta^1; \quad \alpha,\beta = 1,2
\end{aligned} \tag{12}$$

where

$$A_{ij} = A'_{ij} + E_{i4}E_{j4}/E_{44}, \quad A_{i4} = E_{i4}/E_{44}$$

$$B_{ij} = B'_{ij} + E_{i4}F_{j4}/E_{44}, \quad B_{i4} = F_{i4}/E_{44}$$

$$D_{ij} = D'_{ij} + F_{i4}F_{j4}/E_{44}, \quad A_{44} = 1/E_{44}$$

$$A'_{ij}, B'_{ij}, D'_{ij} = \int C'_{ij}(1, z_n, z_n^2)\, dz_n$$

$$E_{i4}, F_{i4} = \frac{1}{h_n}\int C''_{i4}(1, z_n)\, dz_n, \quad E_{44} = \frac{1}{h_n^2}\int S_{44}\, dz_n$$

$$\begin{vmatrix} [L] & [L'] \\ [L'^{T}] & [L''] \end{vmatrix} = \begin{vmatrix} [E] & [F] \\ [F] & [G] \end{vmatrix}^{-1} \tag{13}$$

$$E_{\alpha'\beta'}, F_{\alpha'\beta'}, G'_{\alpha'\beta'} = \int S_{(\alpha'+4)(\beta'+4)}\left(\frac{1}{h_n^2}, \frac{12z_n}{h_n^4}, \frac{144z_n^2}{h_n^6}\right) dz_n; \quad \alpha', \beta' = 1, 2.$$

Strain component displacement relations for sublaminate n

$$\varepsilon_i^0 = u_{i,i}^0, \quad \kappa_i = \psi_{i,i}, \quad \varepsilon_3^0 = u_{1,2}^0 + u_{2,1}^0; \quad \text{(no sum) } i = 1,2$$

$$\kappa_3 = \psi_{1,2} + \psi_{2,1}, \quad \varepsilon_4 = \psi_3, \quad \gamma_\alpha^0 = \psi_\alpha + u_{3,\alpha}^0, \quad \gamma_\alpha^1 = \psi_{3,\alpha}; \quad \alpha = 1, 2. \tag{14}$$

Continuity of displacements and equilibrium of tractions over bonded interfaces (A_n, $n = 2, 3, \ldots, N$)

$$u_i^{0n} + \frac{h_n}{2}\psi_i^n = u_i^{0n-1} - \frac{h_n}{2}\psi_i^{n-1} = \bar{u}_i^n, \quad t_i^{-(n-1)} + t_i^{+n} = 0. \tag{15}$$

Prescribed displacement conditions

$$u_i^{0n} + \frac{h_n}{2}\psi_i^n = u_i^{*+n} \quad n = 1, 2, \ldots, N \text{ over } A_n^*$$

$$u_i^{0n} - \frac{h_n}{2}\psi_i^n = u_i^{*-n} \quad n = 2, 3, \ldots, N+1 \text{ over } A_n^* \tag{16a}$$

$$u_i^{0n} = u_i^{*0n} \quad n = 1, 2, \ldots, N \text{ over } C_u^n$$

$$\psi_i^n = \psi_i^{*n}$$

$$u_i^{*0}, \psi_i^* = \int u_i^*\left(\frac{1}{h_n}, \frac{12z_n}{h_n^3}\right) dz_n. \tag{16b}$$

Prescribed traction conditions on C_σ^n

$$\check{N}_1^* = N_1\nu_1 + N_3\nu_2 - (t_1^{+0} + t_1^{-0})$$

$$\check{N}_2^* = N_3\nu_1 + N_2\nu_2 - (t_2^{+0} + t_2^{-0})$$

$$\check{M}_1^* = M_1\nu_1 + M_3\nu_2 - \frac{h_n}{2}(t_1^{+0} - t_1^{-0})$$

$$\check{M}_2^* = M_3\nu_1 + M_2\nu_2 - \frac{h_n}{2}(t_2^{+0} - t_2^{-0})$$

$$\check{Q}^* = Q_1\nu_1 + Q_2\nu_2 - (t_3^{+0} + t_3^{-0})$$

$$\check{R}^* = R_1\nu_1 + R_2\nu_2 - \frac{h_n}{2}(t_3^{+0} - t_3^{-0}) \tag{17}$$

$$\check{N}_\alpha^*, \check{M}_\alpha^* = \int \check{T}_\alpha^*(1, z_n)\, \mathrm{d}z_n$$

$$\check{Q}^*, \check{R}^* = \int \check{T}_3^*(1, z_n)\, \mathrm{d}z_n. \tag{18}$$

A set of equations similar to eqns (17) with superscript * replaced by 0 relate the unknown stress resultants on C_u^n where eqns (16b) hold. In these equations, and in eqns (17) $t_i^{\pm 0}(C)$ and $t_i^{\pm *}(C)$ are line forces which act on laminate surfaces $z_n = \pm h_n/2$ at contour C. These are either prescribed or unknown. Interactive forces between two sublaminates fall in the latter category. Such forces are expected at the contours which are the free edges of a laminate as discussed later. Similar line forces are contained in $t_i^{\pm}$ in eqns (11) at the contours which are the delamination boundary curves (shown by dotted lines in Fig. 1) across which the displacement components u_i, ψ_i are continuous, but their gradients are not necessarily so. These line forces are caused by the discontinuities in these displacement gradients, and can be called the stress singularities[6–8, 13] in the mixed boundary value problem of bonded sublaminates (or plates) described by eqns (11)–(18). In the next section we will demonstrate the existence of these forces from the exact solution of a sample problem.

In parts of area A, where all the sublaminates are perfectly bonded, the differential eqns (11) can be reduced to a set $3(N+1)$ second-order partial differential equations in terms of $3(N+1)$ displacements at $(N+1)$ surfaces by making use of eqns (15) as well as relations (12). In other parts, where delaminations exist, similar equations are obtained, but the number of unknown interface displacements, as well as equations, will be more. When no displacement conditions are prescribed on surfaces perpendicular to the z-axis, these equations can be solved with enough unknowns, which together with the unknown concentrated forces, can be evaluated to satisfy eqns (16b) and (17), as well as continuity of u_i^n, ψ_i^n and force and moment equilibrium across delamination boundaries.

For example, let us consider the problem to be quasi-3-D and restrict our attention to the plane AA which passes through the tip of a delamination as shown on the top half of Fig. 1. On the right-hand side of AA, there are $3(N+1)$ displacement variables. On the left-hand side, which contains one delamination, there are $3(N+2)$ displacement variables. In the solution of the second-order ordinary differential equations, there will be a total of $12N+18$ unknowns, half of which (i.e. $6N+9$) are available for satisfying displacement continuity and equilibrium conditions across the plane AA and the rest are to be used for satisfying conditions at ends x = constant away from AA. At AA, $3(N+2)$ displacement conditions (three at each of $N+2$ interfaces) and $6N$ equilibrium conditions (six for each of N sublaminates) have to be satisfied. In general, one therefore, needs $3(N-1)$ extra unknowns which come from three concentrated forces at AA in x, y, z direction at each interface other than the top and bottom surfaces of the whole laminate. An exact mathematical proof of the existence of such forces in the general problem is possible but complicated. We will give this proof for a simple problem considered in the next section involving two sublaminates. It is clear, however, that these forces are caused due to the fact that displacements are continuous across AA, but their gradients are not. The existence of such forces at interfaces other than those containing delamination tips may seem impossible at first glance but is a natural consequence of such discretization schemes and assumed fields (similar phenomena are observed in finite element solutions). However, the forces at interfaces other than the ones containing delaminations, will be small and will reduce with increasing N. It is shown in the double cantilever beam problem considered later that even the force at the delamination tip reduces as the number of sublaminates, N, is increased, and the stress field approaches the exact stress solution. As shown in the examples, energy release rates at delamination tips can, however, be evaluated with sufficient accuracy even with a small number of sublaminates.

Pagano[11, 12] obtained a similar approximate formulation for homogeneous sublaminates, and solved the 2-D free edge problem of a laminated composite. In his formulation,

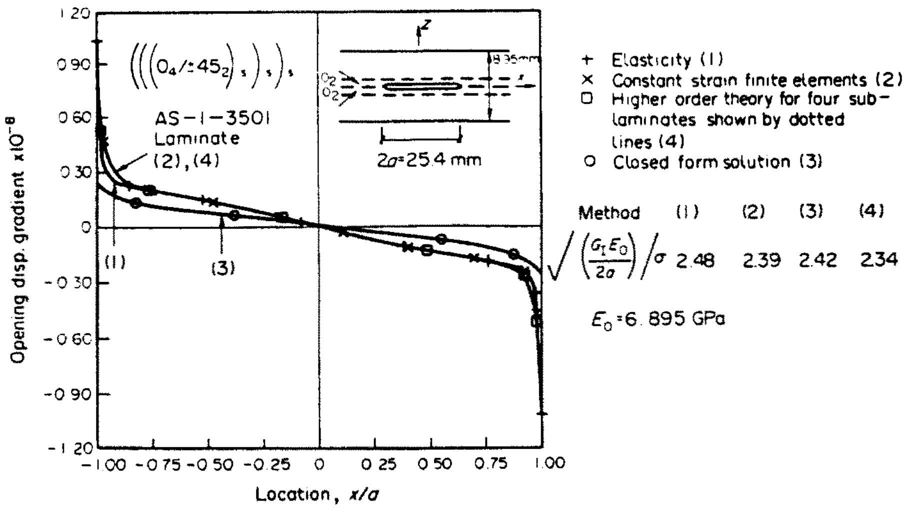


Fig. 2. Results for unit normal stress σ on delamination surfaces in a laminate.

however, the concentrated forces are absent, and the interface stresses at the edges (which are assumed finite) appear to diverge as the number of sublaminates is increased. Even for such 2-D problems, the solution algorithm becomes quite cumbersome as the number of sublaminates is increased. Moreover, a major objective of this investigation was to determine the influence of large deformation and non-linear shear response of adhesives in single lap bonded joints. For these reasons, a finite element solution of such 2-D problems (or quasi-3-D) since three displacements are retained, but all quantities are assumed to be independent of y is formulated as discussed later. Other quasi-3-D problems where stresses are independent of y, but the displacements are not necessarily so (as in the free edge problem of a laminate which is long and loaded in the y-direction) may be obtained by superposition.

STRESS SINGULARITY IN THE FORM OF FORCES

Existence of concentrated forces at disbond tips has been demonstrated for bonded plate models employing shear deformation theory[8, 13]. For the present model, we will demonstrate this with the exact solution for a sample problem. Consider a midplane symmetric laminate which is infinitely long in the x- and y-directions, but contains a disbond of length $2a$ at the midplane (Fig. 2). The disbond surfaces are subjected to a normal pressure distribution (equal and opposite on the two surfaces) which is independent of the y-coordinate so that the problem is quasi-3-D. For simplicity, we will consider the laminate as an assemblage of two sublaminates (one above and the other below the disbond) which are themselves midplane symmetric and balanced. We also assume that $D'_{13} = 0$ which is a good assumption for the case of a repeating layup pattern. The non-vanishing displacements and stress resultants are u_1^0, ψ_1, u_3^0, ψ_3, N_1, M_1, Q_1, N_4, and R_1. Because of symmetry conditions about the disbonded interface we need to consider only the top sublaminate which is stress free at the top surface. We, therefore, have no body forces and

$$\begin{aligned} t_1^+ &= t_1^- = 0 & -\infty < x < \infty \\ t_3^+ &= 0 & -\infty < x < \infty \\ t_3 &= p(x) & -a < x < a. \end{aligned} \qquad (19a\text{–}c)$$

It is clear that one has to solve the first, third, fourth and sixth equation of eqns (11) with second terms and body force terms (f, g) on the left-hand side of each equation equal to zero. The rest of the equations in eqns (11) are identically satisfied.

Writing

$$U_i^{\pm} = u_i^0 \pm h/2\psi_i \quad i = 1, 3 \tag{20}$$

where h is the thickness of the sublaminate, we note that

$$U_3^- = 0, \quad |x| > a. \tag{21}$$

Making use of constitutive relations (12) and eqns (19a)–(19c) one obtains from eqns (11)

$$-(t_1^+ + t_1^-) = \frac{A_{11}}{2}(U_{1,11}^+ + U_{1,11}^-) + \frac{A_{14}}{h}(U_{3,1}^+ - U_{3,1}^-) = 0$$

$$-(t_1^+ + t_1^-)/2 = \frac{D_{11}}{h^2}(U_{1,11}^+ - U_{1,11}^-) - \frac{L_{11}}{h^2}(U_1^+ - U_1^-) - \frac{L_{11}}{2h}(U_{3,1}^+ + U_{3,1}^-) = 0$$

$$-t_3^+ = \left(\frac{L_{11} - A_{14}}{2h}\right)U_{1,1}^+ - \left(\frac{L_{11} + A_{14}}{2h}\right)U_{1,1}^- + \left(\frac{L_{11}}{4} + \frac{L_{11}''}{h^2}\right)U_{3,11}^+$$

$$+\left(\frac{L_{11}}{4} - \frac{L_{11}''}{h^2}\right)U_{3,11}^- - \frac{A_{44}}{h^2}(U_3^+ - U_3^-) = 0$$

$$-t_3^- = \left(\frac{L_{11} + A_{14}}{2h}\right)U_{1,1}^+ - \left(\frac{L_{11} - A_{14}}{2h}\right)U_{1,1}^- + \left(\frac{L_{11}}{4} - \frac{L_{11}''}{h^2}\right)U_{3,11}^+$$

$$+\left(\frac{L_{11}}{4} + \frac{L_{11}''}{h^2}\right)U_{3,11}^- + \frac{A_{44}}{h^2}(U_3^+ - U_3^-). \tag{22a–d}$$

One approach often used in crack problems in elasticity is to employ Fourier transforms and reduce the system of equations to a system of integral equations for determining the displacement discontinuities. We will follow the same procedure and introduce the Fourier transform of a quantity as

$$\bar{t}(s) = \int_{-\infty}^{\infty} t(x')\, e^{isx'}\, dx' \tag{23a–c}$$

where $i = \sqrt{-1}$ and $x' = x/h$, the non-dimensionalized x-coordinate.

Let $a' = a/h$, the non-dimensionalized half disbond length. With the help of known relations between transforms of derivatives of quantities and their transforms and eqns (22a)–(22c) one can express $\bar{U}_1^+$, $\bar{U}_1^-$, and $\bar{U}_3^+$ in terms of $\bar{U}_3^-$. Performing the required algebra and substituting in eqn (22d) the following relation is obtained:

$$\bar{T}_3 = -\frac{1}{l_1 + l_2}\frac{s^2(s^2+\zeta_0^2)}{(s^2+\zeta_1^2)(s^2+\zeta_2^2)}\bar{V}_3 \tag{24a}$$

where $\bar{T}_3$ and $\bar{V}_3$ are Fourier transforms of the quantities

$$T_3(x') = \frac{h^3}{D_{11}} \int_{-\infty}^{x'} t_3^-(x')\, \mathrm{d}x'$$

$$V_3(x') = \frac{1}{h} \frac{\partial U_3^-}{\partial x'} \tag{24b, c}$$

and

$$\begin{aligned} l_1 &= D_{11}/L_{11}h^2 \\ l_2 &= D_{11}/4L''_{11} \\ l_0 &= D_{11}/4(A_{44}-A_{14}^2/A_{11})h^2 \\ \zeta_0^2 &= l_2/l_0 \\ \zeta_1^2+\zeta_2^2 &= (l_1 l_2+l_0)/l_0(l_1+l_2) \\ \zeta_1^2\zeta_2^2 &= l_2/l_0(l_1+l_2). \end{aligned} \tag{25a–f}$$

Because of eqn (21) one must have

$$\int_{-a'}^{a'} V_3(x')\, \mathrm{d}x' = 0. \tag{26}$$

Equations (24) imply that T_3 and V_3 each satisfy fourth-order differential equations for $|x'| > a'$ and $|x'| < a'$, respectively, which may be obtained by considering the two regions of the sublaminate separately. This yields an alternative way for solving the problem by imposing appropriate continuity (of displacements) and equilibrium (of stress resultants) conditions at $|x'| = a'$ with due consideration to concentrated forces at disbond tips. In many cases, such a solution is easier to obtain, and we will follow this procedure in another example considered later. However, to show the existence of these forces, we take the inverse transform of eqn (24a). Noting that $V_3(x') = 0$; $|x'| > a'$ because of (21) one obtains

$$T_3(x') = -\frac{1}{l_1+l_2}\frac{1}{2\pi}\int_{-a'}^{a'} V_3(x'')\int_{-\infty}^{\infty} f(s)\, \mathrm{e}^{is(x''-x')}\, \mathrm{d}s\, \mathrm{d}x'' \tag{27a}$$

where $f(s)$ is the rational function of s^2 in the right-hand side of eqn (24a). In what follows, we will assume that ζ_k^2 are real, and denote ζ_k as their positive square roots. Writing $f(s)$ in the form of partial fractions and making use of some integral formulae[16] eqn (27a) reduces to

$$V_3(x') + \frac{1}{2}\int_{-a'}^{a'}\left[\sum_{k=1}^{2}\frac{b_k}{\zeta_k}\, \mathrm{e}^{-|x''-x'|/\zeta_k}\, V_3(x'')\right]\mathrm{d}x'' = -(l_1+l_2)T_3(x')$$

where

$$\begin{aligned} b_1 &= [\zeta_1^2(l_2^2-l_0)+l_2]/\{l_0(l_1+l_2)(\zeta_1^2-\zeta_2^2)\} \\ b_2 &= -[\zeta_2^2(l_2^2-l_0)+l_2]/\{l_0(l_1+l_2)(\zeta_1^2-\zeta_2^2)\}. \end{aligned} \tag{27b–d}$$

Since $T_3(x')$ is known (see eqns (24b) and (19d)) for $|x'| < a'$, this is a Fredholm integral equation of the second kind for determining V_3 in that domain. We now note that V_3 must be a continuous function for $|x'| < a'$, but there is no other requirement except for condition (26). It may be noted that discontinuous derivatives of displacements are possible if the pressure $p(x)$ is in the form of a concentrated force for $|x'| < a'$ as in shear deformable beam theory.

A solution of the integral equation (which is given later for uniform pressure) will indicate that $V_3(x')$ usually approaches a non-zero value as $|x|' \to a'$ and therefore, because of eqn (21), T_3 has a discontinuity at $x' = a'$ given by

$$T_3(a'^+) - T_3(a'^-) = \frac{V_3(a'^-)}{l_1 + l_2} \tag{28a}$$

where $F(a'^-)$ and $F(a'^+)$ indicate the values of the function as a' is approached from the left and from the right, respectively. A similar discontinuity exists at $x' = -a'$. Therefore, because of eqns (24)

$$t_3^-(x') = \frac{D_{11}}{h^2}\frac{1}{l_1+l_2}[V_3(a'^-)\delta(x'-a') - V_3(-a'^+)\delta(x'+a')] + F(x') \tag{28b}$$

where δ denotes the delta-dirac function and $F(x')$ is a function which is equal to $p(x')$ (the prescribed pressure) for $|x'| < a'$ and for $|x'| > a'$ it is the sum of two exponentially decaying functions, since ζ_k $(k = 1,2)$ are positive and real numbers. It may be noted that for a majority of laminated composites this is true. However, the expressions given above are also valid for complex values of ζ_k^2 (note that $\zeta_2^2 = \bar{\zeta}_1^2$) provided ζ_k are chosen as their roots with positive real parts, so that $\zeta_2 = \bar{\zeta}_1$ and $b_2 = \bar{b}_1$.

We have shown the existence of concentrated forces at disbond tips for a particular problem with only two sublaminates. In a general quasi-3-D problem for an infinitely long laminate with many sublaminates and self-equilibrating tractions on prescribed delamination surfaces or on top and bottom surfaces of the laminates a similar Fourier transform solution is applicable, and a set of integral equations similar to eqn (27b) may be obtained. It may also be shown that the concentrated forces in (t_1, t_2, t_3) at the disbond tips are proportional to the corresponding displacement discontinuity gradients at the same tips, although the constants of proportionality may change depending upon how many sublaminates are used in the model. The general nature of the stress (force) singularity, however, remains the same and will also hold for laminates of finite length. Such forces will also exist at other discontinuities like free edges. In such cases, however, it is better to obtain solutions in a different manner as described earlier, since solutions in addition to those in the form of Fourier transforms must be assumed to satisfy conditions at the ends. Once the concentrated forces t^c and displacement discontinuity gradients (U'^{t}) at the disbond tips are evaluated, the energy release rates may be obtained by the application of Irwin's virtual crack closure method considering another crack tip an infinitesimal distance ahead of the present one. It is easy to show that for the particular singular nature of the stress field, the result for each mode is

$$G = \tfrac{1}{2}t^c U'^{t} \tag{29}$$

where forces and displacements for a particular mode (I–III) are considered. The result is also valid for cracks other than delaminations (e.g. transverse cracks) and is now being widely used in conjunction with standard finite element techniques where t^c is taken as the force transferred at the tip node.

For completeness, we give the solution of eqn (27b) for uniform pressure, and compare the results for a particular laminate with other approximate and rigorous solutions.

When

$$p(x) = p_0$$

$$T_3(x') = p'x', \quad |x'| < a'$$

where

$$p' = p_0 h^3 / D_{11} \tag{30a–d}$$

and an exact solution of eqn (27b) is

$$V_3(x') = \frac{p'x'^3}{6} - C_1 x' + \frac{C_2 \sinh \zeta_0 x'}{\cosh \zeta_0 a'}$$

where C_1 and C_2 are constants chosen to make the coefficients of $\sinh \zeta_k x'$ $(k = 1, 2)$ equal to zero in the expression obtained after substituting eqn (30d) on the left-hand side of eqn (27b) and integrating the second term. For $\tanh \lambda_0 a' \approx 1$ (although it is not necessary to make this assumption) the results can be expressed in the following simple form:

$$C_1 \approx p' \frac{\frac{a'^3}{6} + \frac{(l_3-1)a'^2}{2\zeta_0} - \left\{\frac{(l_3-l_4)}{\zeta_0^2} - l_1\right\}a' - \left\{\frac{l_1+l_2}{\zeta_0^2} - \frac{(l_1 l_2 + l_0)(l_3-1)}{l_2 \zeta_0}\right\}}{a' + (l_3-1)/\zeta_0}$$

$$C_2 \approx p'(l_3 - l_4) \frac{\frac{a'^3}{3} + \frac{l_3}{\zeta_0}a'^2 + \frac{l_3^2}{\zeta_0^2}a' + \frac{l_3 l_1 + l_2}{\zeta_0^2}}{\zeta_0[a' + (l_3-1)/\zeta_0]} \tag{31}$$

where

$$l_3 = (\zeta_1 + \zeta_2)l_1 + l_2$$
$$l_4 = 1 + \zeta_0 l_1 + l_2.$$

From eqns (30d) and (28b) the displacement discontinuity gradients and concentrated forces at $x' = a'$ are

$$U_3^{c} = 2V_3(a') \approx -2(C_1 - C_2 - p'a'^3/6)$$
$$t_3^{c} = \frac{D_{11}}{h^2(l_1+l_2)} V_3(a') \tag{32}$$

and the energy release rate G_1 is obtained from eqn (29). The same result will be obtained from the consideration of change in crack or complementary energy with respect to a. The term crack energy means the work done by the pressure on crack opening displacement and it is often used in the literature on fracture mechanics.

Figure 2 shows the displacement gradients and energy release rates obtained by the following four methods for a 25.4 mm disbond in a 64 ply $((0_4/\pm 45_2/\mp 45_2/0_4)_s)_s$ ASI-3501 plate.

(1) Numerical solution of integral equations to which the exact elasticity problem is reduced[2, 3].

(2) Constant strain finite elements.

(3) Present theory employing four sublaminates (as shown in Fig. 2) with the two sublaminates above and below the delamination containing two 0° layers only. A finite element method discussed later is used for this purpose.

(4) Closed form solution for two sublaminates based on present theory discussed above.

Table 1. Properties used for calculation

AS1-3501-6 Material, ply thickness = 0.1398 mm
0° *Layers*:
$E_x = 125$ GPa, $E_y = E_z = 10$ GPa, $G_{xz} = 5.8$ GPa
$\nu_{xy} = \nu_{xz} = 0.28$, $\nu_{yz} = 0.35$
$\pm 45^\circ$ *Layers (smeared)*:
$E_x = E_y = 19.93$ GPa, $E_z = 11.06$ GPa, $G_{xz} = 4.52$ GPa
$\nu_{xy} = 0.718$, $\nu_{xz} = \nu_{yz} = 0.104$.

A shear correction factor of 5/6 is used in computation of effective shear stiffnesses in the case of one sublaminate model since a repeated lay-up pattern exists[9, 10]. In case more sublaminates are used, the correction factors are omitted.

Properties used for the calculation are given in Table 1. To reduce the number of degrees of freedom for method 2 we have replaced the $\pm 45^\circ$ layers by a homogeneous material with smeared properties (Table 1) based on constant in-plane strain and constant transverse stress assumptions. The same properties are also used in all other methods for the purpose of comparison, although it is not necessary to do so. As can be seen from the figure, the crack opening displacement gradients as well as energy release rates from the first three methods are extremely close to one another. It should be noted, however, that the displacement gradients at the tips are unbounded for the exact elasticity solution, but they are bounded for all other methods. The displacement gradients near the tips obtained from the closed form solution are quite different from those from other methods. Similar differences will also exist in the stress field adhead of the disbond tip (see results for another problem given later). It is clear from the results of method 3, that by choosing thinner sublaminates near delaminations (or increasing the number of sublaminates) more accurate representations are obtained. This phenomena is analogous to that observed in standard finite element solutions with finer mesh size near crack tips. It should be noted, however, that the energy release rate from the closed form solution is extremely close to other solutions. This is not surprising since strength of materials type solutions are often found to yield accurate estimates of energy release rates in many problems. A shear deformation theory solution of this problem is given in Ref. [3].

It should be pointed out that for method 2, it is necessary to model the smeared $\pm 45^\circ$ material and 0° layers separately, whereas for method 3, they are lumped in a sublaminate (except for the 0°_2 sublaminate near the disbonded interface). For this reason the degrees of freedom and the band width for method 2 are about three times those in method 3. It is clear, therefore, that the present method should yield substantial reduction (60%, or more) in computational times.

In the next section we give the outline of finite element modeling using the present approach, and later we will compare the stress fields ahead of a crack tip in another sample problem obtained by increasing the number of sublaminates with those calculated from exact numerical solutions.

FINITE ELEMENT MODEL FOR 2-D PROBLEMS

In finite element modeling, the laminate (or laminated structural component) is divided into a finite number of parts. For problems where some of the sublaminates are truncated before others (as in a bonded joint) each truncation location is made to coincide with one of the boundaries of the partition. Let the nth sublaminate be divided into M parts. The mth element shown in Fig. 1 has length l_m, height h_n and four nodes numbered 1–4. To obtain the local stiffness matrix it is referred to a local coordinate system x', z' (the primes are omitted in the expressions which follow), and it is assumed that the three displacements u_i in the element are given by (the subscripts 1, 2, 3 of u as well as the subscripts m and n to l and h, respectively, are also omitted)

$$u = u^0 + z\psi$$

$$u^0 = \frac{u^1 - u^2 + u^3 + u^4}{4} + \frac{u^3 + u^4 - (u^1 + u^2)}{2l} x$$

$$\psi = \frac{u^1 - u^2 + u^3 - u^4}{2h} + \frac{u^3 - u^4 - (u^1 - u^2)}{lh} x$$

$$u^j = \text{displacement at node } j, j = 1, \ldots, 4. \quad (33)$$

The nodal forces can be easily calculated from values of N, M, Q and R at $x = \pm l/2$ and $t_i^{\pm}$ at $z = \pm h/2$ (which are obtained by the use of eqns (11) so as to satisfy static equilibrium). They could also be obtained by using the virtual work principle. This results in the following relationship between nodal forces and nodal displacements $[\underline{N}] = [K^l][\underline{u}]$ where

$$[\underline{N}]^T = [(N_1^1, N_2^1, N_3^1), (N_1^2, \ldots), (N_1^3, \ldots), (N_1^4, N_2^4, N_3^4)]$$

$$[\underline{u}]^T = [(u_1^1, u_2^1, u_3^1), (u_1^2, \ldots), (u_1^3, \ldots), (u_1^4, u_2^4, u_3^4)]. \quad (34)$$

The superscript denotes the location (node) and the subscript indicates the direction of the forces and displacements and $[K^l]$ is the local (12 × 12) symmetric stiffness matrix which is better written as an assemblage of 16 (3 × 3) submatrices $[K^{\alpha\beta}]$ given in the Appendix. Note that $[K^{\beta\alpha}] = [K^{\alpha\beta}]^{tr}$.

It should be pointed out that if local coordinates of some of the elements are inclined with respect to global x, z coordinates (as in the problem of ply drops) appropriate transformation laws have to be utilized to obtain rotated local stiffness matrices before the global matrix is assembled. In some cases, use of triangular elements are necessary, local stiffness matrices of which are discussed in Ref. [15]. String stiffnesses due to large deformations which should be added to the terms of the stiffness matrix of rectangular elements and modifications necessary to consider the elastoplastic shear stress–strain relation of adhesive elements are also given in Ref. [15].

THE DOUBLE CANTILEVER BEAM PROBLEM

In this section, we consider a wide double cantilever beam shown in Fig. 3 subjected to two equal and opposite forces P on the two arms. We will assume that these forces are the resultants of parabolic shear stress distributions, although any general distribution may be considered without any difficulty. We first give a closed form solution for two sublaminates (one above and the other below the crack) each of which is midplane symmetric and balanced. As in the previous illustrative example, we also assume $D'_{13} = 0$ and consider the top sublaminate only utilizing the symmetry conditions about $z = 0$. However, we will follow a different route by choosing solutions of the differential equations, eqns (11), in the cracked ($-a \leqslant x < 0$) and uncracked parts ($x > 0$) as discussed earlier. We will also assume that the uncracked length is large compared to a and it can be considered to be infinitely long. For brevity, we omit the details and give the expression of non-zero quantities as a function of $x' = x/h$, h is the thickness of the sublaminates, $a' = a/h$ and b is the width of the beam. ζ_0, ζ_1, ζ_2, l_1, l_2, l_0 are the same as in eqns (25).

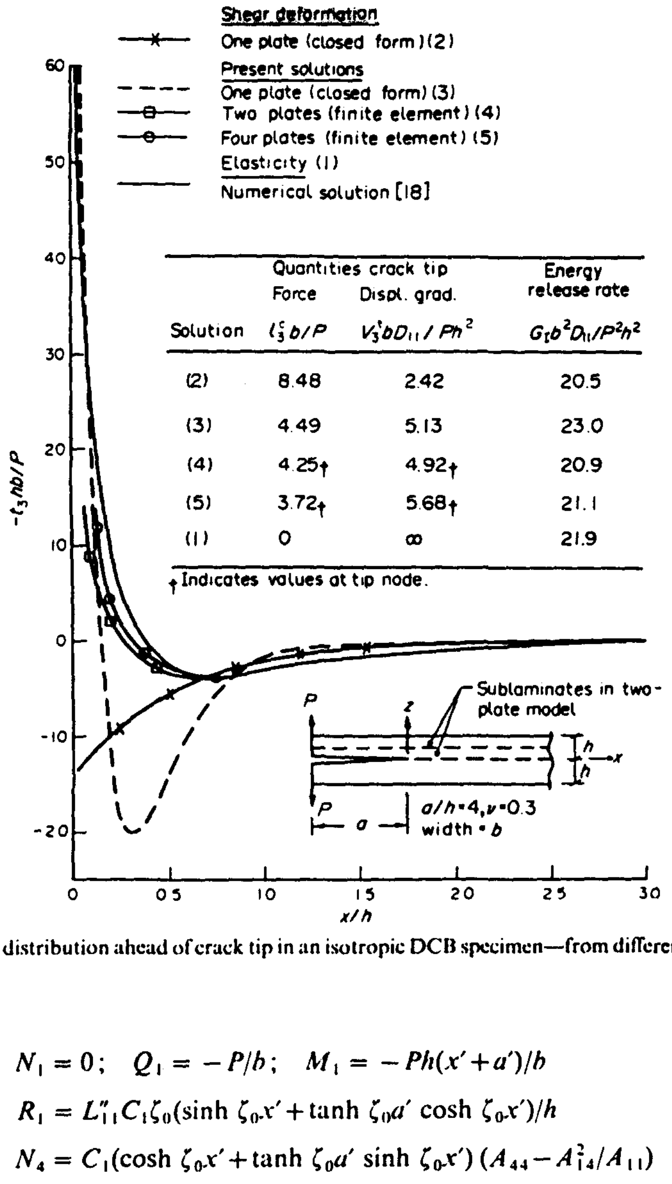
Utilizing the end conditions at $x' = -a'$ ($N_1 = M_1 = R_1 = 0$ and $Q_1 = -P/b$), and the fact that $u_3^0 - \psi_3 h/2 = 0$ at $x' = 0$, the solutions of the equations for $-a' < x' < 0$ are as follows:

$$u_3^0/h = -Px'/bL_{11} + Ph^2x'^3/6bD_{11} + Ph^2a'x'^2/2bD_{11} - C_3x' + C_1/2$$

$$\psi_3 = C_1(\cosh \zeta_0 x' + \tanh \zeta_0 a' \sinh \zeta_0 x') + C_2$$

$$\psi_1 = -Ph^2x'^2/2bD_{11} - Ph^2a'x'/bD_{11} + C_3$$

$$u_1^0/h = C_2 - C_1(\sinh \zeta_0 x' + \tanh \zeta_0 a' \cosh \zeta_0 x')A_{13}/A_{11}\zeta_0$$



Solution	Quantities crack tip Force $t_3^c b/P$	Displ. grad. $V_3' bD_{11}/Ph^2$	Energy release rate $G_I b^2 D_{11}/P^2h^2$
(2)	8.48	2.42	20.5
(3)	4.49	5.13	23.0
(4)	4.25†	4.92†	20.9
(5)	3.72†	5.68†	21.1
(1)	0	∞	21.9

†Indicates values at tip node.

Fig. 3. Stress distribution ahead of crack tip in an isotropic DCB specimen—from different solutions.

$$N_1 = 0; \quad Q_1 = -P/b; \quad M_1 = -Ph(x'+a')/b$$
$$R_1 = L''_{11}C_1\zeta_0(\sinh\zeta_0 x' + \tanh\zeta_0 a' \cosh\zeta_0 x')/h$$
$$N_4 = C_1(\cosh\zeta_0 x' + \tanh\zeta_0 a' \sinh\zeta_0 x')(A_{44} - A_{14}^2/A_{11}) \tag{35}$$

where C_1, C_2, C_3 are arbitrary constants.

All quantities should decay as x' is increased in the solutions for $x' > 0$, which can be written in the following form (B_k, $k = 1, 2$ being arbitrary constants; when ζ_k^2 are complex ζ_k are their roots with positive real parts such that $\zeta_2 = \bar{\zeta}_1$ and $B_2 = \bar{B}_1$)

$$u_3^0/h = \sum_{k=1}^{2} B_k(l_1\zeta_k - 1/\zeta_k)\exp(-\zeta_k x')$$
$$\psi_3 = 2u_3^0/h; \quad \psi_1 = \sum_{k=1}^{2} B_k \exp(-\zeta_k x')$$
$$u_1^0/h = -2A_{14}\left[\sum_{k=1}^{2} B_k(l_1 - 1/\zeta_k^2)\exp(-\zeta_k x')\right]\Big/A_{11}$$
$$N_1 = 0; \quad Q_1 = L_{11}l_1\sum_{k=1}^{2} B_k\zeta_k^2\exp(-\zeta_k x')$$
$$M_1 = -D_{11}\left[\sum_{k=1}^{2} B_k\zeta_k\exp(-\zeta_k x')\right]\Big/h$$

$$R_1 = 2L''_{11}\left[\sum_{k=1}^{2} B_k(l_1\zeta_k^2-1)\exp(-\zeta_k x')\right]\Big/h$$

$$N_4 = 2(A_{14}^2/A_{11}-A_{44})\sum_{k=1}^{2} B_k(l_1\zeta_k-1/\zeta_k)\exp(-\zeta_k x')$$

$$t_3^- = D_{11}\left[\sum_{k=1}^{2} B_k\zeta_k^3\exp(-\zeta_k x')\right]\Big/h^3. \quad (36)$$

Note that $u_3^0-\psi_3 h/2 = 0$.

Denoting the value of the quantity $f(x')$ as $x' \to 0$ from the left and from the right as $f(0^-)$ and $f(0^+)$, respectively, the continuity of displacement components at and equilibrium conditions across $x' = 0$ (with the consideration of a concentrated force t_3^c in the z-direction at the crack tip and taking an infinitesimal element extending on both sides of $x' = 0$) will be satisfied if

$$\psi_1(0^-) = \psi_1(0^+);\quad u_1(0^+) = u_1(0^-);\quad u_3(0^+) = u_3(0^-) \quad (37a\text{–}c)$$

$$M_1(0^-) = M_1(0^+);\quad Q_1(0^+)-Q_1(0^-)+t_3^c = 0;\quad R_1(0^+)-R_1(0^-)-t_3^c h/2 = 0. \quad (38a\text{–}c)$$

Note that there are six unknowns ($C_1, C_2, C_3, B_1, B_2, t_3^c$) to be evaluated from six equations. Whitney[17] obtained a solution to the problem (with minor differences) for the case of a homogeneous beam without any consideration to the concentrated force t_3^c. The equilibrium equation involving R_1, eqn (38c), was left unsatisfied, since there are not enough unknowns without t_3^c. One should note that even if ordinary shear deformation theory is used to solve the problem the concentrated force t_3^c must be considered, otherwise one will be unable to satisfy the equilibrium equation, eqn (38b), involving Q_1. Based on this fact we may conjecture that even with the use of theories of still higher order, concentrated forces will exist in such sublaminate assemblage models. Hopefully, this will be investigated in future studies.

For completeness, the constants obtained from eqns (37a), (37c) and (38a)–(38c) which will be needed later are given below. C_2 may be obtained from eqn (37b)

$$B_1 = -[P^*(1+\zeta_2 a')+t_3^{c*}]/\zeta_1(\zeta_1-\zeta_2)$$

$$B_2 = [P^*(1+\zeta_1 a')+t_3^{c*}]/\zeta_2(\zeta_1-\zeta_2)$$

$$C_3 = B_1+B_2$$

$$t_3^{c*} = -\frac{P^*}{l_1+l_2}\left[\frac{a'(l_3+l_4 f_1)+l_5+l_6 f_1}{l_3 f_1+l_4}\right]$$

$$C_1\zeta_0/2 = (a' l_2\zeta_0+l_2 l_3)/(l_3 f_1+l_4) \quad (39)$$

where l_3, l_4 are given in eqns (31) and

$$P^* = Ph^2/bD_{11}$$

$$t_3^{c*} = t_3^c h^2/D_{11}$$

$$f_1 = \tanh\zeta_0 a'$$

$$l_5 = l_1\zeta_0+l_1+l_2$$

$$l_6 = (\zeta_1+\zeta_2)(l_2+l_2). \quad (40)$$

Other relevant quantities like stress distribution for $x' > 0$ (t_3^-) can be obtained from eqns

Table 2

Theory	Value of C_6
Elasticity[18]	2.34
Present theory	2.77
Shear deformation theory	1.85

(35) and (36). The gradient of the displacement discontinuity $U_3'^{t}$ at the crack tip and the energy release rate are

$$U_3'^{t} = 2 \cdot t_3^{c*}(l_1 + l_2)$$
$$G_1 = \tfrac{1}{2} t_3^c U_3'^{t}. \tag{41}$$

For ordinary shear deformation theory the concentrated force and the displacement discontinuity gradient (denoted with a subscript S) at the tip are

$$t_{3S}^c = P(1+\zeta_3 a')$$
$$V_{3S}'^{t} = t_{3S}^c / L_{11}$$
$$\zeta_3 = (L_{11}h^2/D_{11})^{1/2}. \tag{42}$$

For values of a' such that $f_1 \approx 1$ (see eqns (40)) the energy release rate in eqns (41) can be expressed in the form

$$G_1 = P^2h^2(a'+C_4)^2/b^2D_{11} \tag{43}$$

and a similar expression holds for shear deformation theory C_4 being a constant depending on the theory used.

For an isotropic plate with Poisson's ratio $\nu = 0.3$ and correction factors of 5/6 (with L_{11} and L_{11}'') the stress intensity factor calculated to give the same G_1 can be written in the same form as that for elasticity theory[18] for $a' > 4$

$$K_1 = \frac{P}{b\sqrt{h}}[12a' + C_6] \tag{44}$$

where the values of C_6 are given in Table 2. Although the contribution of C_6 to G_1 (or K_1) is small for large a', this term is absent in Whitney's solution[17]. It should be noted that the energy release rate can also be calculated from

$$G_1 = \frac{P}{b}\frac{\mathrm{d}u_3^0(-a)}{\mathrm{d}a}$$

and the result is the same as in eqns (41) or eqns (42) although the algebra is quite complex to show this equivalence in the case of eqns (41).

Figure 3 gives the comparison of the stress distribution ahead of the crack tip obtained for the isotropic case calculated from five solutions.

(1) Elasticity solution[18].
(2) Closed form solution (shear deformation theory).
(3) Closed form solution (present formulation), one sublaminate for $z > 0$.
(4) Finite element solution based on present formulation with two sublaminates used to model the half $z > 0$.
(5) Same as above with four sublaminates.

For methods 4 and 5 we have not used any correction factors, although 5/6 is utilized for 2 and 3. Also given in this figure are the values of concentrated forces and displacement gradients at the tip obtained from methods 2–5 which show that the concentrated forces reduce and the gradients increase gradually as more sublaminates (or higher order theory) are used. This is consistent with the fact that in the case of the exact solution, the displacement gradients become unbounded. It should be noted, however, that the products of the force and the gradients for all cases are roughly the same, since energy release rates are usually not sensitive to the model used. It is seen, however, that the stress distribution ahead of the tip obtained from the present formulation approach the elasticity solution as the number of sublaminates is increased.

CONCLUDING REMARKS

The mixed variational principle and the approximate formulation in terms of an assemblage of sublaminates are shown to provide a powerful, yet relatively inexpensive computational algorithm for approximate stress analysis in composite laminates in presence of delaminations. Non-zero values of displacement gradients and concentrated forces at disbond tips are shown to be the exact mathematical forms of singular fields in the solution of mixed boundary value problems using the sublaminate assemblage model. Obviously, they differ from the singular fields in exact elasticity solutions, but better representations of such fields are obtained by increasing the number of sublaminates. It is also shown that energy release rates can be obtained from the local field at crack tips, which is very useful in complicated problems. The finite element model described is for quasi-3-D problems and can be extended to 3-D problems with a little effort and such extensions will result in substantial reduction in computer time required for 3-D analyses as compared to standard finite element methods where layers of different orientations have to be modeled separately. Results for other delamination problems computed with this model have been compared extensively in Ref. [15] with those from other finite element methods modeling each layer, as well as from numerical solutions based on rigorous elasticity theory. Applications to the problem of stress analysis near a ply-termination and fracture mechanics analysis of bonded joints have also been demonstrated in Ref. [15]. The stiffness matrix of triangular elements required in transition regions for ply drop problems can be found in Ref. [15]. It should be pointed out that in 3-D or 2-D problems of delaminations between two dissimilar isotropic or anisotropic layers, oscillatory type stress singularities are often observed which sometimes result in part closure of disbonds as discussed in Refs [1, 3, 5, 8]. In the present formulation, such oscillatory singularities are absent, but part closures are still possible, and should be expected if the interactive concentrated force at a tip is compressive in nature. The extent of such part closures may be determined if an appropriate algorithm is devised for this purpose. Further studies in this direction will be helpful for obtaining a better understanding of the effect of interlaminar stress fields on fatigue and fracture behavior on laminated composite components.

Acknowledgements—The authors express their sincere appreciation to the Naval Air Development Center for sponsoring this work under Contract No. N-62269-82-C-0705.

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APPENDIX: ELEMENTS OF SUBMATRICES OF LOCAL STIFFNESS MATRIX

$$[K^{11}] = \begin{bmatrix} a_{11}+2p_{11} & b_{11}+2p_{12} & -a_{13}-a_{31} \\ & d_{11}+2p_{22} & -d_{13}-d_{31} \\ \text{Sym.} & & e_{11}+2p_{33} \end{bmatrix}$$

$$[K^{12}] = \begin{bmatrix} a_{12}-2p_{11} & b_{12}-2p_{12} & a_{13}-a_{32} \\ b_{21}-2p_{12} & d_{12}-2p_{22} & d_{13}-d_{32} \\ a_{31}-a_{23} & d_{31}-d_{23} & e_{12}-2p_{33} \end{bmatrix}$$

$$[K^{13}] = \begin{bmatrix} -a_{11}+p_{11} & -b_{11}+p_{12} & a_{31}-a_{13} \\ -b_{11}+p_{12} & -d_{11}+p_{22} & d_{31}-a_{13} \\ a_{13}-a_{31} & d_{13}-d_{31} & -e_{11}+p_{33} \end{bmatrix}$$

$$[K^{14}] = \begin{bmatrix} -a_{12}-p_{11} & -b_{12}-p_{12} & a_{13}+a_{32} \\ -b_{21}-p_{12} & -d_{12}-p_{22} & d_{13}+d_{32} \\ a_{31}+a_{23} & d_{31}+d_{23} & -e_{12}-p_{33} \end{bmatrix}$$

$$[K^{22}] = \begin{bmatrix} a_{22}+2p_{11} & b_{22}+2p_{12} & a_{23}+a_{32} \\ & d_{22}+2p_{22} & d_{23}+d_{32} \\ \text{Sym.} & & e_{22}+2p_{33} \end{bmatrix}$$

$$[K^{23}] = \begin{bmatrix} -a_{12}-p_{11} & -b_{21}-p_{12} & -a_{31}-a_{23} \\ -b_{12}-p_{12} & -d_{12}-p_{22} & -d_{31}-d_{23} \\ -a_{32}-a_{13} & -d_{32}-d_{13} & -e_{12}-p_{33} \end{bmatrix}$$

$$[K^{24}] = \begin{bmatrix} -a_{22}+p_{11} & -b_{22}+p_{12} & a_{23}-a_{32} \\ -b_{22}+p_{12} & -d_{22}+p_{22} & d_{23}-d_{32} \\ a_{32}-a_{23} & d_{32}-d_{23} & -e_{22}+p_{33} \end{bmatrix}$$

$$[K^{33}] = \begin{bmatrix} a_{11}+2p_{11} & b_{11}+2p_{12} & a_{13}+a_{31} \\ & d_{11}+2p_{22} & d_{13}+d_{31} \\ \text{Sym.} & & e_{11}+2p_{33} \end{bmatrix}$$

$$[K^{34}] = \begin{bmatrix} a_{12}-2p_{11} & b_{12}-2p_{12} & a_{32}-a_{13} \\ b_{21}-2p_{12} & d_{12}-2p_{22} & d_{32}-d_{13} \\ a_{23}-a_{31} & d_{23}-d_{31} & e_{12}-2p_{33} \end{bmatrix}$$

$$[K^{44}] = \begin{bmatrix} a_{22}+2p_{11} & b_{22}+2p_{12} & -a_{23}-a_{32} \\ & d_{22}+2p_{22} & -d_{23}-d_{32} \\ \text{Sym.} & & e_{22}+2p_{33} \end{bmatrix}$$

where

$$a_{11} = \left(\frac{A_{11}}{4} + \frac{B_{11}}{h} + \frac{D_{11}}{h^2}\right)\Big/ l, \quad a_{12} = \left(\frac{A_{11}}{4} - \frac{D_{11}}{h^2}\right)\Big/ l$$

$$a_{22} = \left(\frac{A_{11}}{4} - \frac{B_{11}}{h} + \frac{D_{11}}{h^2}\right)\Big/ l, \quad a_{13} = \frac{A_{14}}{4h} + \frac{B_{14}}{2h^2}, \quad a_{23} = \frac{A_{14}}{4h} - \frac{B_{14}}{2h^2}$$

$$b_{11} = \left(\frac{A_{13}}{4} + \frac{B_{13} + B_{31}}{2h} + \frac{D_{13}}{h^2}\right)\Big/ l, \quad b_{12} = \left(\frac{A_{13}}{4} - \frac{B_{13} - B_{31}}{2h} - \frac{D_{13}}{h^2}\right)\Big/ l$$

$$b_{21} = \left(\frac{A_{13}}{4} + \frac{B_{13} - B_{31}}{2h} - \frac{D_{13}}{h^2}\right)\Big/ l, \quad b_{22} = \left(\frac{A_{13}}{4} - \frac{B_{13} + B_{31}}{h} + \frac{D_{13}}{h^2}\right)\Big/ l$$

$$d_{13} = \frac{A_{34}}{4h} + \frac{B_{34}}{2h^2}, \quad d_{23} = \frac{A_{34}}{4h} - \frac{B_{34}}{2h^2}$$

$$d_{11} = \left(\frac{A_{33}}{4} + \frac{B_{33}}{h} + \frac{D_{33}}{h^2}\right)\Big/ l, \quad d_{12} = \left(\frac{A_{33}}{4} - \frac{D_{33}}{h^2}\right)\Big/ l$$

$$d_{22} = \left(\frac{A_{33}}{4} - \frac{B_{33}}{h} + \frac{D_{33}}{h^2}\right)\Big/ l$$

$$a_{31} = \frac{L_{11}}{4h} + \frac{L'_{11}}{2h^2}, \quad d_{31} = \frac{L_{12}}{4h} + \frac{L'_{21}}{2h^2}$$

$$a_{32} = \frac{L_{11}}{4h} - \frac{L'_{11}}{2h^2}, \quad d_{32} = \frac{L_{12}}{4h} - \frac{L'_{21}}{2h^2}$$

$$e_{11} = \left(\frac{L_{11}}{4} + \frac{L'_{11}}{h} + \frac{L''_{11}}{h^2}\right)\Big/ l, \quad e_{12} = \left(\frac{L_{11}}{4} - \frac{L''_{11}}{h^2}\right)\Big/ l$$

$$e_{22} = \left(\frac{L_{11}}{4} - \frac{L'_{11}}{h} + \frac{L''_{11}}{h^2}\right) l, \quad p_{33} = A_{44}l/6h^2$$

$$p_{\alpha\beta} = L_{\alpha\beta}l/6h^2; \quad \alpha, \beta = 1, 2.$$